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EXTENSIONS OF THE HADAMARD DETERMINANT THEOREM

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ABSTRACT
Denote by H_n the set of n by n, positive definite hermitian matrices. Hadamard proved that $h(A) \geqq \det(A)$ for all $A \in H_n$, where $h(A)$ is the product of the main diagonal elements of A. Subsequently, M. Marcus showed that $\operatorname{per}(A) \geqq h(A)$ for all $A \in H_n$. This article contains a result for all generalized matrix functions from which it follows that $h(A) \geqq (\operatorname{per}(A^{1/n}))^n$, $A \in H_n$.

Denote by H_n the convex cone of n by n positive definite hermitian matrices. Let G be a permutation group of degree n. Suppose χ is an irreducible, complex character of G. If $A = (a_{ij})$ is an n by n matrix, define

$$d(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}.$$

For example, if $G = S_n$ and $\chi = \varepsilon$, $d(A) = \det(A)$. If $G = S_n$ and $\chi = 1$, $d(A) = \operatorname{per}(A)$, the permanent of A. At the other end of the line, when $G = \{e\}$, $d(A)$ is Hadamard's function,

$$h(A) = \prod_{t=1}^{n} a_{tt}.$$

The Hadamard determinant theorem asserts that $h(A) \geqq \det(A)$, for all $A \in H_n$. In 1918, I. Schur obtained the following vast improvement: $d(A)/\chi(e) \geqq \det(A)$, for all $A \in H_n$. In 1963, M. Marcus proved a "Hadamard theorem for permanents", namely, $h(A) \leqq \operatorname{per}(A)$, for all $A \in H_n$. In this note we provide another extension of Hadamard's theorem.

THEOREM. *Let G be a subgroup of S_n. Suppose χ is an irreducible, complex character of G. Then*

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$$h(A) \geqq [d(A^{1/n})/\chi(e)]^n \tag{1}$$

for all $A \in H_n$.

Before proving this result, it may be appropriate to make some observations. If $G = S_n$ and $\chi = \varepsilon$, then $d(A^{1/n})/\chi(e) = \det(A^{1/n}) = (\det A)^{1/n}$. Thus, (1) reduces to Hadamard's Theorem. If $G = S_n$, $\chi = 1$, the result becomes

$$h(A) \geqq (\operatorname{per}(A^{1/n}))^n. \tag{2}$$

In [4], it was shown that

$$f(t) = (\operatorname{per}(A^t))^{1/t}$$

is a strictly increasing function of t for all $A \in H_n$. In [9], P. J. Nikolai extended this result to any d-function for which $\chi(e) = 1$. In fact (see [6, lemma 2]), Nikolai's methods apply to arbitrary irreducible characters. They yield, with an appropriate definition at $t = 0$, that

$$[d(A^t)/\chi(e)]^{1/t} \tag{3}$$

is a nondecreasing function of the real variable t, for all $A \in H_n$. It follows that $h(A)$ dominates (3) for any $t(\neq 0)$ less than $1/n$.

I briefly entertained the notion that $h(A)$ might dominate (3) for any $t \leqq \frac{1}{2}$ $(t \neq 0)$, regardless of the value of n. That this is false can be seen from the following example. Note that $h(A) \geqq (\operatorname{per}(A^{1/2}))^2$ for all $A \in H_n$ if and only if

$$h(A^2) \geqq (\operatorname{per} A)^2,$$

for all $A \in H_n$. Let

$$S = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 0 & -1 & 3 \end{bmatrix} \in H_6.$$

Then $h(S^2) = 19 \cdot 12^5 = 4{,}727{,}808$ whereas $(\operatorname{per} S)^2 = (2{,}259)^2 = 5{,}103{,}081$.

Finally, it is clear from the Theorem and Marcus's result that $\operatorname{per}(A^n) \geqq (\operatorname{per} A)^n$, $A \in H_n$. This observation is easily eclipsed, however, by the following consequence of the Cauchy–Schwarz inequality (see [8, p. 25]): $\operatorname{per}(A^2) \geqq (\operatorname{per} A)^2$, $A \in H_n$.

PROOF OF THEOREM. We use a device of A. W. Marshall and I. Olkin. (Also see [1].) Let $A_i = D_iAD_i$, $1 \leqq i \leqq 2^n$, where the D_i are distinct diagonal matrices of the form $D_i = \operatorname{diag}(\varepsilon_1, \cdots, \varepsilon_n)$, with $\varepsilon_j = \pm 1$, $j = 1, 2, \cdots, n$. Note that

$$\begin{aligned} d(A_i) &= \sum_{\sigma \in G} \chi(\sigma) \prod_{t=1}^{n} \varepsilon_t a_{t\sigma(t)} \varepsilon_{\sigma(t)} \\ &= \sum_{\sigma \in G} \chi(\sigma) \left(\prod_{t=1}^{n} \varepsilon_t^2 \right) \left(\prod_{t=1}^{n} a_{t\sigma(t)} \right) \\ &= d(A). \end{aligned}$$

Further, since D_iAD_i is a unitary similarity of A, $(D_iAD_i)^s = D_iA^sD_i$. For $0 < s$, define $d^s : H_n \to \mathbf{R}^+$ by $d^s(A) = d(A^s)$. Then, in summary

$$\begin{aligned} d^s(A_i) &= d(D_iA^sD_i) \\ &= d(A^s) \\ &= d^s(A). \end{aligned}$$

Marshall and Olkin [5] observed that $\Sigma 2^{-n} A_i = \operatorname{diag}(a_{11}, a_{22}, \cdots, a_{nn})$. It is proved in [7] that d^s is concave on H_n for $0 < s \leqq 1/n$. Therefore,

$$\begin{aligned} \chi(e) h(A)^s &= d^s(\operatorname{diag}(a_{11}, a_{22}, \cdots, a_{nn})) \\ &= d^s\left(\sum 2^{-2} A_i \right) \\ &\geqq \sum 2^{-n} d^s(A_i) \\ &= d^s(A) \\ &= d(A^s). \end{aligned}$$

■

If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A \in H_n$, it has been shown [8, p. 113] that

$$\operatorname{per}(A^{1/n}) \leqq \frac{1}{n} \sum_{i=1}^{n} \lambda_i. \tag{4}$$

It was pointed out to me by C. R. Johnson that (2) is better than (4). By the arithmetic-geometric mean inequality, $h(A) \leqq ((\operatorname{trace} A)/n)^n$.

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